

The Hankel Transform of the Sum of Consecutive Generalized Catalan Numbers

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Abstract. *We discuss the properties of the Hankel transformation of a sequence whose elements are the sums of consecutive generalized Catalan numbers and find their values in the closed form.*

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1. INTRODUCTION

The *Hankel transform* of a given sequence $A = \{a_0, a_1, a_2, \dots\}$ is the sequence of Hankel determinants $\{h_0, h_1, h_2, \dots\}$ (see Layman [7]) where $h_n = |a_{i+j-2}|_{i,j=1}^n$, i.e

$$A = \{a_n\}_{n \in \mathbb{N}_0} \quad \rightarrow \quad h = \{h_n\}_{n \in \mathbb{N}_0} : \quad h_n = \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & & a_{n+1} \\ \vdots & & \ddots & \\ a_n & a_{n+1} & & a_{2n} \end{vmatrix} \quad (1)$$

In this paper, we will consider the sequence of the sums of two adjacent generalized Catalan numbers with parameter L :

$$a_0 = L + 1, \quad a_n = a_n(L) = c(n; L) + c(n + 1; L) \quad (n \in \mathbb{N}), \quad (2)$$

where

$$c(n; L) = T(2n, n; L) - T(2n, n - 1; L), \quad (3)$$

with

$$T(n, k; L) = \sum_{j=0}^{n-k} \binom{k}{j} \binom{n-k}{j} L^j. \quad (4)$$

Example 1.1. Let $L = 1$. Vandermonde's convolution identity implies that

$$\binom{n}{k} = \sum_j \binom{k}{j} \binom{n-k}{j}.$$

Hence

$$T(2n, n; 1) = \binom{2n}{n}, \quad T(2n, n-1; 1) = \binom{2n}{n-1},$$

wherefrom we get Catalan numbers

$$c(n) = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1} \binom{2n}{n}$$

and

$$a_n = c(n) + c(n+1) = \frac{(2n)!(5n+4)}{n!(n+2)!} \quad (n = 0, 1, 2, \dots).$$

In the paper [3], A. Cvetković, P. Rajković and M. Ivković have proved that the Hankel transform of a_n equals sequence of Fibonacci numbers with odd indices

$$h_n = F_{2n+1} = \frac{1}{\sqrt{5} 2^{n+1}} \left\{ (\sqrt{5} + 1)(3 + \sqrt{5})^n + (\sqrt{5} - 1)(3 - \sqrt{5})^n \right\}.$$

Example 1.2. For $L = 2$ we get like $a_n(2)$ the next numbers

$$3, 8, 28, 112, 484, \dots,$$

and the Hankel transform h_n :

$$3, 20, 272, 7424, 405504, \dots$$

One of us, P. Barry conjectured that

$$h_n(2) = 2^{\frac{n^2-n}{2}-2} \left\{ (2 + \sqrt{2})^{n+1} + (2 - \sqrt{2})^{n+1} \right\}.$$

In general, P. Barry made the conjecture, which we will prove through this paper.

Theorem 1.1. (The main result) *For the generalized Pascal triangle associated to the sequence $n \mapsto L^n$, the Hankel transform of the sequence*

$$c(n; L) + c(n+1; L)$$

is given by

$$h_n = \frac{L^{(n^2-n)/2}}{2^{n+1} \sqrt{L^2 + 4}} \cdot \left\{ (\sqrt{L^2 + 4} + L)(\sqrt{L^2 + 4} + L + 2)^n + (\sqrt{L^2 + 4} - L)(L + 2 - \sqrt{L^2 + 4})^n \right\}. \quad (5)$$

From now till the end, let us denote by

$$\xi = \sqrt{L^2 + 4}, \quad t_1 = L + 2 + \xi, \quad t_2 = L + 2 - \xi. \quad (6)$$

Now, we can write

$$h_n = \frac{L^{n(n-1)/2}}{2^{n+1}\xi} \cdot ((\xi + L)t_1^n + (\xi - L)t_2^n).$$

Or, introducing

$$\varphi_n = t_1^n + t_2^n, \quad \psi_n = t_1^n - t_2^n \quad (n \in \mathbb{N}_0), \quad (7)$$

the final statement can be expressed by

$$h_n = \frac{L^{n(n-1)/2}}{2^{n+1}\xi} \cdot (L\psi_n + \xi\varphi_n). \quad (8)$$

Lemma 1.1. *The values φ_n and ψ_n satisfy the next relations*

$$\varphi_j \cdot \varphi_k = \varphi_{j+k} + (4L)^j \varphi_{k-j}, \quad \psi_j \cdot \psi_k = \varphi_{j+k} - (4L)^j \varphi_{k-j} \quad (0 \leq j \leq k) \quad (9)$$

$$\varphi_j \cdot \psi_k = \psi_{j+k} + (4L)^j \psi_{k-j}, \quad \psi_j \cdot \varphi_k = \psi_{j+k} - (4L)^j \psi_{k-j} \quad (0 \leq j \leq k). \quad (10)$$

Corollary 1.1. *Assuming that the main theorem is true, the function $h_n = h_n(L)$ is the next polynomial*

$$h_n(L) = 2^{-n} L^{n(n-1)/2} \cdot \left\{ \sum_{i=0}^{[(n-1)/2]} \binom{n}{2i+1} L(L+2)^{n-2i-1} (L^2+4)^i + \sum_{i=0}^{[n/2]} \binom{n}{2i} (L+2)^{n-2i} (L^2+4)^i \right\}.$$

Proof. By previous notation, we can write

$$\begin{aligned} & (L + \xi)(L + 2 + \xi)^n - (L - \xi)(L + 2 - \xi)^n \\ &= (L + \xi) \sum_{k=0}^n \binom{n}{k} (L + 2)^{n-k} \xi^k - (L - \xi) \sum_{k=0}^n (-1)^k \binom{n}{k} (L + 2)^{n-k} \xi^k \\ &= \sum_{k=0}^n (1 - (-1)^k) \binom{n}{k} L(L + 2)^{n-k} \xi^k + \sum_{k=0}^n (1 + (-1)^k) \binom{n}{k} (L + 2)^{n-k} \xi^{k+1} \\ &= 2 \sum_{i=0}^{[(n-1)/2]} \binom{n}{2i+1} L(L + 2)^{n-2i-1} \xi^{2i+1} + 2 \sum_{i=0}^{[n/2]} \binom{n}{2i} (L + 2)^{n-2i} \xi^{2i+1} \\ &= 2\xi \left\{ \sum_{i=0}^{[(n-1)/2]} \binom{n}{2i+1} L(L + 2)^{n-2i-1} \xi^{2i} + \sum_{i=0}^{[n/2]} \binom{n}{2i} (L + 2)^{n-2i} \xi^{2i} \right\}, \end{aligned}$$

wherefrom immediately follows the polynomial expression for h_n . \square

2. THE GENERATING FUNCTION FOR THE SEQUENCES OF NUMBERS AND ORTHOGONAL POLYNOMIALS

The Jacobi polynomials are given by

$$P_n^{(a,b)}(x) = \frac{1}{2^n} \sum_{k=0}^n \binom{n+a}{k} \binom{n+b}{n-k} (x-1)^{n-k} (x+1)^k \quad (a, b > -1).$$

Also, they can be written in the form

$$P_n^{(a,b)}(x) = \left(\frac{x-1}{2}\right)^n \sum_{k=0}^n \binom{n+a}{k} \binom{n+b}{n-k} \left(\frac{x+1}{x-1}\right)^k.$$

From the fact

$$L = \frac{x+1}{x-1} \quad \Leftrightarrow \quad x = \frac{L+1}{L-1} \quad (x \neq 1, L \neq 1).$$

we conclude that

$$\begin{aligned} T(2n, n; L) &= (L-1)^n \cdot P_n^{(0,0)}\left(\frac{L+1}{L-1}\right), \\ T(2n+2, n; L) &= (L-1)^n \cdot P_n^{(2,0)}\left(\frac{L+1}{L-1}\right). \end{aligned}$$

The generating function $G(x, t)$ for the Jacobi polynomials is

$$G^{(a,b)}(x, t) = \sum_{n=0}^{\infty} P_n^{(a,b)}(x) t^n = \frac{2^{a+b}}{\phi \cdot (1-t+\phi)^a \cdot (1+t+\phi)^b}, \quad (11)$$

where

$$\phi = \phi(x, t) = \sqrt{1 - 2xt + t^2}.$$

Now,

$$\begin{aligned} \sum_{n=0}^{\infty} T(2n, n; L) t^n &= \sum_{n=0}^{\infty} P_n^{(0,0)}\left(\frac{L+1}{L-1}\right) ((L-1)t)^n = G^{(0,0)}\left(\frac{L+1}{L-1}, (L-1)t\right), \\ \sum_{n=0}^{\infty} T(2n+2, n; L) t^n &= \sum_{n=0}^{\infty} P_n^{(2,0)}\left(\frac{L+1}{L-1}\right) ((L-1)t)^n = G^{(2,0)}\left(\frac{L+1}{L-1}, (L-1)t\right). \end{aligned}$$

Also,

$$\begin{aligned} \sum_{n=0}^{\infty} T(2n, n-1; L) t^n &= t \cdot \left\{ G^{(2,0)}\left(\frac{L+1}{L-1}, (L-1)t\right) - 1 \right\}, \\ \sum_{n=0}^{\infty} T(2n+2, n+1; L) t^n &= \frac{1}{t} \cdot \left\{ G^{(0,0)}\left(\frac{L+1}{L-1}, (L-1)t\right) - 1 \right\}. \end{aligned}$$

The generating function $\mathcal{G}(t; L)$ for the sequence $\{a_n\}_{n \geq 0}$ is given by

$$\begin{aligned} \mathcal{G}(t; L) &= \sum_{n=0}^{\infty} a_n t^n \\ &= \frac{t+1}{t} G^{(0,0)}\left(\frac{L+1}{L-1}, (L-1)t\right) - (t+1) G^{(2,0)}\left(\frac{L+1}{L-1}, (L-1)t\right) - \frac{1}{t}. \end{aligned} \quad (12)$$

After some computation, we prove the next theorem.

Theorem 2.1. *The generating function $\mathcal{G}(t; L)$ for the sequence $\{a_n\}_{n \geq 0}$ is*

$$\mathcal{G}(t; L) = \frac{t+1}{\rho(t; L)} \left\{ \frac{1}{t} - \frac{4}{(1 - (L-1)t + \rho(t; L))^2} \right\} - \frac{1}{t}, \quad (13)$$

where

$$\rho(t; L) = \phi\left(\frac{L+1}{L-1}, (L-1)t\right) = \sqrt{1 - 2(L+1)t + (L-1)^2 t^2} \quad (14)$$

The function $\rho(t; L)$ has domain

$$D_\rho = \left(-\infty, \frac{1 - 2\sqrt{L} + L}{1 - 2L + L^2}\right) \cup \left(\frac{1 + 2\sqrt{L} + L}{1 - 2L + L^2}, +\infty\right) \quad (L \neq 1),$$

and

$$D_\rho = (-\infty, 1/4) \quad (L = 1).$$

Example 2.1. For $L = 1$, we get

$$\mathcal{G}(t; 1) = \sum_{n=0}^{\infty} a_n(1) t^n = \frac{1}{t} \left(\frac{(1 - \sqrt{1-4t})(1+t)}{2t} - 1 \right). \quad (15)$$

and for $L = 2$, we find

$$\mathcal{G}(t; 2) = \sum_{n=0}^{\infty} a_n(2) t^n = -\frac{1}{t} + \frac{t+1}{\sqrt{t^2 - 6t + 1}} \left\{ \frac{1}{t} - \frac{4}{(1 - t + \sqrt{t^2 - 6t + 1})^2} \right\}. \quad (16)$$

3. THE WEIGHT FUNCTION CORRESPONDING TO THE FUNCTIONAL

It is known (for example, see Krattenthaler [6]) that the Hankel determinant h_n of order n of the sequence $\{a_n\}_{n \geq 0}$ equals

$$h_n = a_0^n \beta_1^{n-1} \beta_2^{n-2} \cdots \beta_{n-2}^2 \beta_{n-1}, \quad (17)$$

where $\{\beta_n\}_{n \geq 1}$ is the sequence given by:

$$\mathcal{G}(x) = \sum_{n=0}^{\infty} a_n x^n = \frac{a_0}{1 + \alpha_0 x - \frac{\beta_1 x^2}{1 + \alpha_1 x - \frac{\beta_2 x^2}{1 + \alpha_2 x - \cdots}}} \quad (18)$$

The sequences $\{\alpha_n\}_{n \geq 0}$ and $\{\beta_n\}_{n \geq 1}$ are the coefficients in the recurrence relation

$$Q_{n+1}(x) = (x - \alpha_n)Q_n(x) - \beta_n Q_{n-1}(x), \quad (19)$$

where $\{Q_n(x)\}_{n \geq 0}$ is the monic polynomial sequence orthogonal with respect to the functional \mathcal{U} determined by

$$\mathcal{U}[x^n] = a_n \quad (n = 0, 1, 2, \dots). \quad (20)$$

In this section the functional will be constructed for the sum of consecutive generalized Catalan numbers.

We would like to express $\mathcal{U}[f]$ in the form:

$$\mathcal{U}[f(x)] = \int_R f(x) d\psi(x),$$

where $\psi(x)$ is a distribution, or, even more, to find the weight function $w(x)$ such that $w(x) = \psi'(x)$.

Denote by $F(z; L)$ the function

$$F(z; L) = \sum_{k=0}^{\infty} a_k z^{-k-1}.$$

From the generating function (13), we have:

$$F(z; L) = z^{-1} \mathcal{G}(z^{-1}; L). \quad (21)$$

and after some simplifications we obtain that

$$\begin{aligned} F(z; L) &= -1 + \frac{2(z+1)}{L-1+z+\sqrt{L^2+(z-1)^2-2L(z+1)}} \\ &= -1 + \frac{2(z+1)}{L-1+z(1+z\rho(\frac{1}{z}, L))} \end{aligned}$$

Example 3.1. From (15) and (16), we yield

$$\begin{aligned} F(z; 1) &= z^{-1} \mathcal{G}(z^{-1}; 1) = \frac{1}{2} \left\{ z - 1 - (z+1) \sqrt{1 - \frac{4}{z}} \right\}, \\ F(z; 2) &= \frac{-1}{2z} \left\{ 1 + z \left(2 - z + (z+1) \sqrt{1 - \frac{6}{z} + \frac{1}{z^2}} \right) \right\}. \end{aligned}$$

Notice that

$$\begin{aligned} \int F(z; 2) dz &= z + \frac{1}{4} z(z-1) \rho(1/z, 2) + \log(z) \\ &\quad - \frac{1}{2} \log \left(1 + z(\rho(1/z, 2) - 3) \right) - \frac{7}{2} \log(z - 3 + z\rho(1/z, 2)). \end{aligned}$$

It will be the impulse for further discussion.

Denote by

$$R(z; L) = z\rho(\frac{1}{z}, L) = \sqrt{L^2 + (z-1)^2 - 2L(z+1)}.$$

From the theory of distribution functions (see Chihara [2]), especially by the Stieltjes inversion formula

$$\psi(t) - \psi(0) = -\frac{1}{\pi} \lim_{y \rightarrow 0^+} \int_0^t \Im F(x + iy; L) dx, \quad (22)$$

we conclude that holds

$$\mathcal{F}(z; L) = \int F(z; L) dz = \frac{1}{4} \left[z^2 - 2Lz - (z - L + 1)R(z; L) - l_1(z) + l_2(z) \right], \quad (23)$$

where

$$\begin{aligned} l_1(z) &= 2(3L + 1) \log \left[z - (L + 1) + R(z; L) \right] \\ l_2(z) &= 2(L - 1) \log \left[\frac{-(L - 1)R(z; L) - (L - 1)^2 + z(L + 1)}{z^2(L - 1)^3} \right] \end{aligned}$$

Rewriting the function $R(z; L)$ in the form

$$R(z; L) = \sqrt{(z - L - 1)^2 - 4L}$$

and replacing $z = x + iy$, we have

$$R(x; L) = \lim_{y \rightarrow 0^+} R(x + iy; L) = \begin{cases} i\sqrt{4L - (x - L - 1)^2}, & x \in (a, b); \\ \sqrt{(x - L - 1)^2 - 4L}, & \text{otherwise,} \end{cases}$$

where

$$a = (\sqrt{L} - 1)^2, \quad b = (\sqrt{L} + 1)^2. \quad (24)$$

In the case when $x \notin ((\sqrt{L} - 1)^2, (\sqrt{L} + 1)^2)$, value $R(x; L)$ is real. Therefore we can calculate imaginary part of $\mathcal{F}(x; L) = \lim_{y \rightarrow 0^+} \mathcal{F}(x + iy; L)$:

$$\Im \mathcal{F}(x; L) = \Im [l_2(x) - l_1(x)] = 0.$$

Otherwise, if $x \in ((\sqrt{L} - 1)^2, (\sqrt{L} + 1)^2)$ we have that:

$$\begin{aligned} l_1(x) &= 2(3L + 1) \log \left[x - (L + 1) \pm i\sqrt{4L - (x - L - 1)^2} \right] \\ \Im l_1(x) &= \begin{cases} 2(3L + 1) \arctan \frac{\sqrt{4L - (x - L - 1)^2}}{x - (L + 1)}, & x \geq L + 1; \\ 2(3L + 1) \left(\pi + \arctan \frac{\sqrt{4L - (x - L - 1)^2}}{x - (L + 1)} \right), & x < L + 1 \end{cases} \\ l_2(x) &= 2(L - 1) \log \left[\frac{-(L - 1)^2 + 2x(L + 1) - i(L - 1)\sqrt{4L - (x - L - 1)^2}}{x^2(L - 1)^3} \right] \\ \Im l_2(x) &= \begin{cases} 2(L - 1) \left(2\pi + \arctan \frac{x(L + 1) - (L - 1)^2}{\sqrt{4L - (x - L - 1)^2}} \right), & x \geq \frac{(L - 1)^2}{L + 1}; \\ 2(L - 1) \left(\pi + \arctan \frac{x(L + 1) - (L - 1)^2}{\sqrt{4L - (x - L - 1)^2}} \right), & x < \frac{(L - 1)^2}{L + 1} \end{cases} \end{aligned}$$

After substituting all considered cases in (23), we finally obtain the value

$$\Im \mathcal{F}(x; L) = \lim_{y \rightarrow 0^+} \Im \mathcal{F}(x + iy; L) = \Im l_2(x) - \Im l_1(x) - (x - L + 1)\sqrt{4L - (x - L - 1)^2}$$

From the relation (22), we conclude that

$$\omega(x; L) = \psi'(x; L) = -\frac{1}{\pi} \frac{d}{dx} \Im \mathcal{F}(x; L) \quad (25)$$

and finally, we obtain

$$\omega(x; L) = \frac{1}{2\pi} \left(1 + \frac{1}{x}\right) \sqrt{4L - (x - L - 1)^2} = \frac{\sqrt{L}}{\pi} \left(1 + \frac{1}{x}\right) \sqrt{1 - \left(\frac{x - L - 1}{2\sqrt{L}}\right)^2} \quad (26)$$

Previous formula holds for $x \in (a, b)$, and otherwise is $\omega(x; L) = 0$.

4. DETERMINING THE THREE-TERM RECURRENCE RELATION

The crucial moment in our proof of the conjecture is to determine the sequence of polynomials $\{Q_n(x)\}$ orthogonal with respect to the weight $w(x; L)$ given by (26) on the interval (a, b) and to find the sequences $\{\alpha_n\}$ $\{\beta_n\}$ in the three-term recurrence relation.

Example 4.1. For $L = 4$, we can find the first members

$$\begin{aligned} Q_0(x) &= 1, & \|Q_0\|^2 &= 5, \\ Q_1(x) &= x - \frac{24}{5}, & \|Q_1\|^2 &= \frac{104}{5}, \\ Q_2(x) &= x^2 - \frac{127}{13}x + \frac{256}{13}, & \|Q_2\|^2 &= \frac{1088}{13}, \\ Q_3(x) &= x^3 - \frac{541}{17}x^2 + \frac{1096}{17}x - \frac{1344}{17}, & \|Q_3\|^2 &= \frac{5696}{17}, \end{aligned}$$

wherefrom

$$\alpha_0 = \frac{24}{5}, \quad \beta_0 = 5, \quad \alpha_1 = \frac{323}{65}, \quad \beta_1 = \frac{104}{25}, \quad \alpha_2 = \frac{1104}{221}, \quad \beta_2 = \frac{680}{169}.$$

Hence

$$h_1 = a_0 = 5, \quad h_2 = a_0^2 \beta_1 = 104, \quad h_3 = a_0^3 \beta_1^2 \beta_2 = 5^3 \left(\frac{104}{25}\right)^2 \frac{680}{169} = 8704.$$

At the beginning, we will notice that in the definition of the weight function appears the square root member.

That's why, let us consider the monic orthogonal polynomials $\{S_n(x)\}$ with respect to the $p^{(1/2, 1/2)}(x) = \sqrt{1 - x^2}$ on the interval $(-1, 1)$. These polynomials are monic Chebyshev polynomials of the second kind:

$$S_n(x) = \frac{\sin((n+1) \arccos x)}{2^n \cdot \sqrt{1 - x^2}}$$

They satisfy the three-term recurrence relation (Chihara [2]):

$$S_{n+1}(x) = (x - \alpha_n^*) S_n(x) - \beta_n^* S_{n-1}(x) \quad (n = 0, 1, \dots), \quad (27)$$

with initial values

$$S_{-1}(x) = 0, \quad S_0(x) = 1,$$

where

$$\alpha_n^* = 0 \quad (n \geq 0) \quad \text{and} \quad \beta_0^* = \frac{\pi}{2}, \quad \beta_n^* = \frac{1}{4} \quad (n \geq 1).$$

If we use the weight function $\hat{w}(x) = (x - c) p^{(1/2, 1/2)}(x)$, then the corresponding coefficients $\hat{\alpha}_n$ and $\hat{\beta}_n$ can be evaluated as follows (see, for example, Gautschi [4])

$$\begin{aligned}\lambda_n &= S_n(c), \\ \hat{\alpha}_n &= c - \frac{\lambda_{n+1}}{\lambda_n} - \beta_{n+1}^* \frac{\lambda_n}{\lambda_{n+1}}, \\ \hat{\beta}_n &= \beta_n^* \frac{\lambda_{n-1} \lambda_{n+1}}{\lambda_n^2} \quad (n \in \mathbb{N}_0).\end{aligned}\tag{28}$$

From the relation (27), we conclude that the sequence $\{\lambda_n\}_{n \in \mathbb{N}}$ satisfies the following recurrence relation:

$$4\lambda_{n+1} - 4c\lambda_n + \lambda_{n-1} = 0 \quad (\lambda_{-1} = 0; \lambda_0 = 1).\tag{29}$$

The characteristic equation

$$4z^2 - 4c z + 1 = 0$$

has the solutions

$$z_{1,2} = \frac{1}{2} \left(c \pm \sqrt{c^2 - 1} \right).$$

and the integral solution of (29) is

$$\lambda_n = E_1 z_1^n + E_2 z_2^n \quad (n \in \mathbb{N}).$$

We evaluate values E_1 and E_2 from the initial conditions ($\lambda_{-1} = 0; \lambda_0 = 1$).

In order to solve our problem, we will choose $c = -\frac{L+2}{2\sqrt{L}}$. Hence

$$z_k = \frac{-t_k}{4\sqrt{L}} \quad (k = 1, 2), \quad \text{where} \quad t_{1,2} = L + 2 \pm \sqrt{L^2 + 4}.$$

Finally, we obtain:

$$\lambda_n = \frac{(-1)^n}{2 \cdot 4^n L^{\frac{n}{2}} \sqrt{L^2 + 4}} \left(t_1^{n+1} - t_2^{n+1} \right) \quad (\lambda = -1, 0, 1, \dots),$$

i.e.,

$$\lambda_n = \frac{(-1)^n}{2 \cdot 4^n L^{\frac{n}{2}} \xi} \psi_{n+1} \quad (\lambda = -1, 0, 1, \dots).$$

After replacing in (28), we obtain:

$$\hat{\alpha}_n = -\frac{L+2}{2\sqrt{L}} + \frac{1}{4\sqrt{L}} \cdot \frac{\psi_{n+2}}{\psi_{n+1}} + \sqrt{L} \cdot \frac{\psi_{n+1}}{\psi_{n+2}},\tag{30}$$

$$\hat{\beta}_n = \frac{\psi_n \psi_{n+2}}{4\psi_{n+1}^2}.\tag{31}$$

If a new weight function $\tilde{w}(x)$ is introduced by

$$\tilde{w}(x) = \hat{w}(ax + b)$$

then we have

$$\tilde{\alpha}_n = \frac{\hat{\alpha}_n - b}{a}, \quad \tilde{\beta}_n = \frac{\hat{\beta}_n}{a^2} \quad (n \geq 0).$$

Now, by using $x \mapsto \frac{x-L-1}{2\sqrt{L}}$, i.e., $a = \frac{1}{2\sqrt{L}}$ and $b = -\frac{L+1}{2\sqrt{L}}$, we have the weight function

$$\tilde{w}(x) = \hat{w}\left(\frac{x-L-1}{2\sqrt{L}}\right) = \frac{1}{2} \left(\frac{x-L-1}{2\sqrt{L}} + \frac{L+2}{2\sqrt{L}} \right) \sqrt{1 - \left(\frac{x-L-1}{2\sqrt{L}} \right)^2}.$$

Thus

$$\tilde{\alpha}_n = -1 + \frac{1}{2} \cdot \frac{\psi_{n+2}}{\psi_{n+1}} + 2L \cdot \frac{\psi_{n+1}}{\psi_{n+2}} \quad (n \in \mathbb{N}_0), \quad (32)$$

and

$$\tilde{\beta}_0 = (L+2)\frac{\pi}{2}, \quad \tilde{\beta}_n = L \frac{\psi_n \psi_{n+2}}{\psi_{n+1}^2} \quad (n \in \mathbb{N}). \quad (33)$$

Example 4.2. For $L = 4$, we get

$$\begin{aligned} P_0(x) &= 1, & \|P_0\|^2 &= 3\pi, \\ P_1(x) &= x - \frac{17}{3}, & \|P_1\|^2 &= \frac{32\pi}{3}, \\ P_2(x) &= x^2 - \frac{43}{4}x + \frac{101}{4}, & \|P_2\|^2 &= 42\pi, \\ P_3(x) &= x^3 - \frac{331}{21}x^2 + \frac{1579}{21}x - \frac{2189}{21}, & \|P_3\|^2 &= \frac{3520\pi}{21}, \end{aligned}$$

wherefrom

$$\tilde{\alpha}_0 = \frac{17}{3}, \quad \tilde{\beta}_0 = 3\pi, \quad \tilde{\alpha}_1 = \frac{61}{12}, \quad \tilde{\beta}_1 = \frac{32}{9}, \quad \tilde{\alpha}_2 = \frac{421}{84}, \quad \tilde{\beta}_2 = \frac{63}{16}.$$

Introducing the weight

$$\check{w}(x) = \frac{2L}{\pi} \tilde{w}(x)$$

will not change the monic polynomials and their recurrence relations, only it will multiply the norms by the factor $2L/\pi$, i.e.

$$\check{P}_k(x) \equiv P_k(x), \quad \|\check{P}_k\|_{\check{w}}^2 = \int_a^b \check{P}_k(x) \check{w}(x) dx = \frac{2L}{\pi} \|P_k\|^2 \quad (k \in \mathbb{N}_0),$$

$$\check{\beta}_0 = L(L+2), \quad \check{\beta}_k = \tilde{\beta}_k \quad (k \in \mathbb{N}), \quad \check{\alpha}_k = \tilde{\alpha}_k \quad (k \in \mathbb{N}_0).$$

Here is

$$\check{\beta}_0 \check{\beta}_1 \cdots \check{\beta}_{n-1} = \frac{L^n}{2} \cdot \frac{\psi_{n+1}}{\psi_n}. \quad (34)$$

In the book [5], W. Gautschi has treated the next problem: If we know all about the MOPS orthogonal with respect to $\check{w}(x)$ what can we say about the sequence $\{Q_n(x)\}$ orthogonal with respect to a weight

$$w_d(x) = \frac{\check{w}(x)}{x-d} \quad (d \notin \text{support}(\tilde{w})) ?$$

W. Gautshi has proved that, by the auxiliary sequence

$$r_{-1} = - \int_{\mathbb{R}} w_d(x) dx, \quad r_n = d - \check{\alpha}_n - \frac{\check{\beta}_n}{r_{n-1}} \quad (n = 0, 1, \dots),$$

it can be determined

$$\begin{aligned} \alpha_{d,0} &= \check{\alpha}_0 + r_0, & \alpha_{d,k} &= \check{\alpha}_k + r_k - r_{k-1}, \\ \beta_{d,0} &= -r_{-1}, & \beta_{d,k} &= \check{\beta}_{k-1} \frac{r_{k-1}}{r_{k-2}} \quad (k \in \mathbb{N}). \end{aligned}$$

In our case it is enough to take $d = 0$ to get the final weight

$$w(x) = \frac{\check{w}(x)}{x}.$$

Hence

$$r_{-1} = -(L+1), \quad r_n = -\left(\check{\alpha}_n + \frac{\check{\beta}_n}{r_{n-1}}\right) \quad (n = 0, 1, \dots). \quad (35)$$

Lemma 4.1. *The parameters r_n have the explicit form*

$$r_n = -\frac{\psi_{n+1}}{\psi_{n+2}} \cdot \frac{L\psi_{n+2} + \xi\varphi_{n+2}}{L\psi_{n+1} + \xi\varphi_{n+1}} \quad (n \in \mathbb{N}_0). \quad (36)$$

Proof. We will use the mathematical induction. For $n = 0$, we really get the expected value

$$r_0 = -\frac{L^2 + 2L + 2}{(L+1)(L+2)}.$$

Suppose that it is true for $k = n$. Now, by the properties for φ_n and ψ_n , we have

$$\check{\alpha}_{n+1} \cdot r_n + \check{\beta}_{n+1} = -\frac{\psi_{n+1}}{\psi_{n+3}} \cdot \frac{L\psi_{n+3} + \xi\varphi_{n+3}}{L\psi_{n+1} + \xi\varphi_{n+1}}.$$

Dividing with r_n , we conclude that the formula is valid for r_{n+1} . \square

Example 4.3. For $L = 4$, we get

$$r_{-1} = -5, \quad r_0 = -\frac{13}{15}, \quad r_1 = -\frac{51}{52}, \quad r_2 = -\frac{356}{357},$$

wherefrom

$$\alpha_0 = \frac{24}{5}, \quad \beta_0 = 5, \quad \alpha_1 = \frac{323}{65}, \quad \beta_1 = \frac{104}{25}, \quad \alpha_2 = \frac{1104}{221}, \quad \beta_2 = \frac{680}{169},$$

just the same as in the Example 4.1.

Proof of the main result. The Krattenthaler's formula (17) can be also written in the form

$$h_1 = a_0, \quad h_n = \beta_0\beta_1\beta_2 \cdots \beta_{n-2}\beta_{n-1} \cdot h_{n-1}. \quad (37)$$

From the theory of orthogonal polynomials, it is known that

$$\|Q_{n-1}\|^2 = \beta_0\beta_1\beta_2 \cdots \beta_{n-2}\beta_{n-1} \quad (n = 2, 3, \dots), \quad (38)$$

wherefrom

$$h_1 = a_0, \quad h_n = \|Q_{n-1}\|^2 \cdot h_{n-1} \quad (n = 2, 3, \dots). \quad (39)$$

Here,

$$\|Q_{n-1}\|^2 = \beta_0 \frac{r_{n-2}}{r_{-1}} \prod_{k=0}^{n-2} \check{\beta}_k = \frac{L^{n-1}}{2} \cdot \frac{L\psi_n + \xi\varphi_n}{L\psi_{n-1} + \xi\varphi_{n-1}}. \quad (40)$$

We will apply the mathematical induction again. The formula for h_n is true for $n = 1$. Suppose that it is valid for $k = n - 1$. Then

$$h_n = \frac{L^{n-1}}{2} \cdot \frac{L\psi_n + \xi\varphi_n}{L\psi_{n-1} + \xi\varphi_{n-1}} \cdot \frac{L^{(n-1)(n-2)/2}}{2^n \xi} \cdot (L\psi_{n-1} + \xi\varphi_{n-1}),$$

wherefrom it follows that the final statement

$$h_n = \frac{L^{n(n-1)/2}}{2^{n+1}\xi} \cdot (L\psi_n + \xi\varphi_n) \quad (n \in \mathbb{N})$$

is true. \square

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